J Eng Math (2008) 60:43–53
DOI 10.1007/s10665-007-9182-2

Interaction of a characteristic shock with a weak discontinuity in a relaxing gas

J. Jena · V. D. Sharma

Received: 28 September 2004 / Accepted: 26 July 2007 / Published online: 1 September 2007

Abstract The evolution of a characteristic shock in a relaxing gas is investigated and its interaction with a weak discontinuity is studied. A particular solution to the governing system, which exhibits space–time dependence, is used to study the evolutionary behaviour of the characteristic shock; the properties of incident, reflected and transmitted waves, influenced by the relaxation mechanism, together with the geometry of the fluid flow and the background state at the rear of the shock, are studied.

Keywords Characteristic shock · Nonlinear wave interaction · Relaxing gases · Weak discontinuities

1 Introduction

It is well known that, at high temperatures, the internal modes of gas molecules get excited and, as a result, there is a transfer of energy from one internal mode to another, until the equilibrium between these modes is re-established. For instance, in the supersonic flight of projectiles at high altitude, or when the gas is compressed by the mechanical action of a piston or by the passage of a shock front, temperatures of many thousands of degrees Kelvin can easily be attained; in this process, the whole energy goes initially to increase the translational energy, and it is followed by a relaxation from translational mode to the rotational mode and from translational mode to vibrational mode until the equilibrium between these modes is re-established. Indeed, the vibrational modes, which unlike translational and rotational modes relax for a longer duration, substantially influence the flow behaviour. Thus, it is essential that at high temperatures the non-equilibrium effects must be considered in the fluid-dynamic equations; but then the analysis becomes considerably more complex than the classical gas-dynamic flow because of (i) nonlinear coupling between the relaxing mode and the fluid flow, and (ii) non-existence of the self-similar nature of the classical flow field. It may be remarked that the various energy modes of air at meteorological temperatures get excited; the vibrational energy content of the air has nevertheless a dominant effect on the sound absorption and on the structure of nonlinear waves (see [1–6]). Apart from this, the system governing the motion of a relaxing gas possesses several

J. Jena
Department of Mathematics, NSIT, Dwarka, New Delhi, India
e-mail: jjena@nsit.ac.in

V. D. Sharma (✉)
Department of Mathematics, IIT Bombay, Mumbai, India
e-mail: vsharma@math.iitb.ac.in

features that make it worth for further study (see, for instance, [7, pp. 236–237] and the references therein). This, indeed serves as a great motivating factor to look for new insight into the behaviour of the wave pattern that finally develops under the influence of a rate process, included in the system of equations governing the motion. Here, instead of working with complicated actual air composition, we use a simple relaxing gas model which brings out all the important features without unduly complicating the calculations. The quantity $Q(p, \rho, \sigma)$, which appears in the governing system of equations and denotes the rate of change of vibrational energy σ, is a known function. The situation $Q = 0$ corresponds to a physical process involving no relaxation; indeed, it includes both the cases in which the vibrational mode is either inactive or follows the translational mode according as the flow is either frozen ($\sigma =$ constant) or in equilibrium ($\sigma = \bar{\sigma}$), where $\bar{\sigma}$ is the equilibrium value of σ evaluated at local p and ρ.

As the relaxation mechanism makes the interaction problem considerably cumbersome, we felt that a simple analysis applicable to planar and radially symmetric motions, where we can use special solutions, would be of interest to elucidate the effects of relaxation on the wave patterns, that finally develop. As the governing system with relaxation is still hyperbolic, in order to study the shock interaction problem, we look for a particular solution of the governing system through which the shock propagates, and investigate the effects of initial discontinuities associated with the incident wave, the geometry of the fluid flow together with the background state at the rear of shock, and the relaxation mechanism present in the flow on the evolutionary behaviour of a characteristic shock and the reflected and transmitted waves after the collision. To our knowledge, such an analytical and numerical description of a complete history of evolution of a characteristic shock has not been studied previously. This paper brings out some interesting features of the evolutionary behaviour of waves in a relaxing gas.

2 Basic equations

The governing equations can be written in the matrix form [1, 2],

$$U_t + AU_x = f, \tag{1}$$

where $U = (\rho, u, p, \sigma)^{\text{tr}}$, $f = (-m\rho u/x, 0, -((\gamma - 1)\rho Q + m\gamma p u/x), Q)^{\text{tr}}$, and

$$A = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 1/\rho & 0 \\ 0 & \rho a^2 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}.$$

Here, x is the distance, t the time, ρ the density, u the particle velocity, p the pressure, σ the vibrational energy, γ the frozen specific-heat ratio in the gas and $a = (\gamma p/\rho)^{1/2}$ the frozen speed of sound. A subscript denotes partial differentiation with respect to the indicated variable unless stated otherwise, and the superscript tr denotes transposition. The system (1) represents a flow which is planar, cylindrically symmetric or spherically symmetric according as $m = 0$, 1 or 2, respectively. The quantity $Q = (\bar{\sigma} - \sigma)/\tau$ denotes the rate of change of vibrational energy, where $\bar{\sigma} = \sigma_0 + c(\rho\rho_0)^{-1}(p\rho_0 - \rho p_0)$ and the suffix 0 refers to an initial equilibrium reference state; the quantities τ and c which are, respectively, the relaxation time and the ratio of vibrational specific heat to the specific gas constant, are assumed to be constant. The equation of state is taken to be of the form $p = \rho R T$.

3 Characteristic shock

For a characteristic shock, the shock surface coincides with a characteristic surface and its velocity coincides with an eigenvalue of the system, both ahead and behind the shock (see [8], [9, pp. 23–26], [10, pp. 74–75], [11, p. 43]). A sufficient condition for the existence of a characteristic shock is that the corresponding eigenvalue is exceptional (linearly degenerate), i.e., $\nabla\lambda^{(i)} \cdot R^{(i,k)} = 0$ where $R^{(i,k)}, k = 1, \ldots, m_i$, denote the right eigenvectors corresponding to the eigenvalue $\lambda^{(i)}$ with m_i its multiplicity and ∇ the gradient operator with respect to the field vector

U. If the corresponding eigenvalue is multiple (i.e., $m_i > 1$), then the shock is characteristic as the eigenvalue is necessarily linearly degenerate (see ([9, pp. 23–36)], [12, pp. 85–86]). The evolution of characteristic shocks in different material media has been studied by Anile et al. [10], Virgopia and Ferraioli [13] and Jena and Sharma [14]. The matrix A in (1) has eigenvalues

$$\lambda^{(1)} = (u + a), \quad \lambda^{(2)} = u \text{ (a double root)}, \quad \lambda^{(3)} = (u - a), \tag{2}$$

with the corresponding left and right eigenvectors

$$\begin{aligned}
&L^{(1)} = (0, \rho a, 1, 0), && R^{(1)} = (1/(2a^2), 1/(2\rho a), 1/2, 0)^{\text{tr}},\\
&L^{(2,1)} = (-a^2, 0, 1, 0), && R^{(2,1)} = (-a^{-2}, 0, 0, 0)^{\text{tr}},\\
&L^{(2,2)} = (0, 0, 0, 1), && R^{(2,2)} = (0, 0, 0, 1)^{\text{tr}},\\
&L^{(3)} = (0, -\rho a, 1, 0), && R^{(3)} = (1/(2a^2), -1/(2\rho a), 1/2, 0)^{\text{tr}}.
\end{aligned} \tag{3}$$

The multiplicity of the eigenvalue $\lambda^{(2)} = u$ indicates that there exists a characteristic shock propagating with the speed $V = u$. Since, across a characteristic shock no mass flow takes place, the Rankine–Hugoniot conditions across this shock can be given by $[u] = 0$, $[p] = 0$, $[\rho] = \zeta$ and $[\sigma] = \eta$, where ζ and η are unknown functions of t to be determined; here $[X]$, defined as $[X] = X - X_*$, denotes the jump in X across the characteristic shock, where X_* and X, on the right-hand side, are the values just ahead of the shock and behind the shock, respectively. The evolutionary law for ζ and η can be obtained by multiplying (1) by the eigenvectors $L^{(2,1)}$ and $L^{(2,2)}$, respectively; and then one obtains, on forming the jumps across the characteristic shock in the usual manner,

$$L\frac{\mathrm{d}[U]}{\mathrm{d}t} + [L]\frac{\mathrm{d}U_*}{\mathrm{d}t} = L[f] + [L]f_* \tag{4}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + u\partial/\partial x$ denotes the material derivative following the shock. Now, using (1) and (3) in (4), we obtain the following transport equations for the quantities ζ and η

$$\begin{aligned}
\frac{\mathrm{d}\zeta}{\mathrm{d}t} &= -\left(\frac{mu}{x} + u_x\right)\zeta + \frac{(\gamma-1)(\rho-\zeta)}{\gamma\tau p}\{\zeta(\sigma_0 - cp_0/\rho_0 - \sigma + \eta) - \rho\eta\},\\
\frac{\mathrm{d}\eta}{\mathrm{d}t} &= -\frac{1}{\tau}\left\{\eta + \frac{cp\zeta}{\rho(\rho-\zeta)}\right\}.
\end{aligned} \tag{5}$$

3.1 Particular case

Let us consider the case when the flow behind the characteristic shock is described by the following equations

$$u(x,t) = k(t)x, \quad \rho = \rho(t), \quad p = p(t), \quad \sigma = \sigma(t), \tag{6}$$

where the particle velocity exhibits linear dependence on the spatial coordinate; indeed, such a state can be visualized in terms of an atmosphere filled with a gas which has spatially uniform pressure variations on account of the particle motion and the spatially uniform relaxation rate. This class of solutions of the governing system has been discussed by Pert [15], Sharma et al. [16] and Clarke [17]; Pert showed that such a form of the velocity distribution is useful in modelling the free expansion of polytropic gases, and it is attained in the limit of large time.

In view of (6), Eqs. $(1)_1$ and $(1)_2$ yield on integration the following forms of the parameters ρ and k

$$\rho = \rho_0(1 + (t - t_0)k_0)^{-(m+1)}, \quad k(t) = k_0/\{1 + (t - t_0)k_0\}, \tag{7}$$

whereas Eqs. $(1)_3$ and $(1)_4$ lead to the following system of ordinary differential equations (ODEs) in p and σ

$$\begin{aligned}
&\frac{\mathrm{d}p}{\mathrm{d}t} + \left(\gamma(m+1)k + \frac{(\gamma-1)c}{\tau}\right)p - \frac{(\gamma-1)}{\tau}\rho\sigma + \frac{(\gamma-1)}{\tau}\rho\,(\sigma_0 - cp_0/\rho_0) = 0,\\
&\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{cp}{\tau\rho} + \frac{\sigma}{\tau} - \frac{1}{\tau}\left(\sigma_0 - \frac{cp_0}{\rho_0}\right) = 0,
\end{aligned} \tag{8}$$

where k_0 and ρ_0 refer to the initial reference state. Using the dimensionless variables

$$\tilde{\zeta} = \zeta/\rho_0, \quad \tilde{\rho} = \rho/\rho_0, \quad \tilde{p} = p/p_0, \quad \tilde{\eta} = \eta\rho_0/p_0, \quad \tilde{\sigma} = \sigma\rho_0/p_0,$$
$$\tilde{t} = t/t_0, \quad \tilde{\tau} = \tau/t_0, \quad \tilde{k} = kt_0, \quad \tilde{k_0} = k_0 t_0, \quad \tilde{\sigma_0} = \sigma_0\rho_0/p_0,$$

and then suppressing the tilde sign (for the sake of convenience), we can write the evolutionary equations (5) in the following form

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = -(m+1)k\zeta + \frac{(\gamma-1)(\rho-\zeta)}{\gamma\tau p}\{\zeta(\sigma_0 - c - \sigma + \eta) - \rho\eta\},$$
$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{1}{\tau}\left\{\eta + \frac{cp\zeta}{\rho(\rho-\zeta)}\right\}, \tag{9}$$

where $\rho = (k_0(t-1)+1)^{-(m+1)}$, $k = k_0/(k_0(t-1)+1)$ and p and σ solve the system (8).

The evolutionary behaviour of the characteristic shock influenced by the relaxation effects, the background state behind the shock and geometrical spreading of the fluid flow, which enter through the parameters c, τ, k_0 and m, can be studied by solving the system of ODEs (9) together with (8) and the initial conditions $p|_{t=1} = p_0, \sigma|_{t=1} = \sigma_0, \zeta|_{t=1} = \zeta_0$ and $\eta|_{t=1} = \eta_0$. The parameter c is of order one or less; in particular for air, it is of order 10^{-1} to 10^{-3}, whilst the parameter τ covers the whole range from very small to very large (see [2,5]). The computations of ζ and η have been carried out for plane ($m = 0$), cylindrically symmetric ($m = 1$) and spherically symmetric ($m = 2$) flows by taking $\gamma = 1.4$ and the values of c and τ lying in the range $0.005 \leq c \leq 0.7$ and $5 \leq \tau \leq 15$; Figs. 1 and 2 show the evolution of a characteristic shock influenced by the physical parameters c, τ, m and k_0. It is observed that an increase in c or τ^{-1} causes ζ and η to decrease, showing thereby that the characteristic shock is progressively smoothened out under the diffusive effect of relaxation present in the medium (see, Figs. 1(a), 1(b), 2(a) and 2(b)). We also note that there is an extra attenuation owing to geometrical spreading and flow variations behind the shock in the sense that an increase in m or k_0 causes ζ and η to decrease (see, Figs. 1(c), 1(d), 2(c) and 2(d)). It is possible to establish the small- and large-time behaviour of some of the variables using standard asymptotic analysis; indeed, for $\sqrt{\beta} < \alpha/2$, where α and β are positive quantities defined as $\beta = \frac{\gamma(m+1)k_0}{\tau}$ and $\alpha = \frac{1+c(\gamma-1)}{\tau} + \gamma(m+1)k_0$, we find from (8) and (9) that $\rho \sim \rho_0$, $k \sim k_0$, $p \sim p_0 + O((t-1)^2)$, $\sigma \sim \sigma_0 + O((t-1)^2)$, $\zeta \sim \zeta_0\{1 - \frac{(t-1)^2}{2\gamma}\beta\}$ and $\eta \sim \eta_0\{1 - \frac{(t-1)^2}{2\gamma}\beta\}$ as $t \to 1$, showing thereby that in the neighbourhood of $t = 1$, both ζ and η decrease from their initial values ζ_0 and η_0. The numerical results, depicted in Figs. (1)–(2), are in conformity with the above-mentioned asymptotic results. Similarly, when $\sqrt{|\kappa|} < |\delta|/2$ where $\kappa = (\gamma-1)(c-\tau\triangle)/(\gamma\tau^2)$ and $\delta = (\gamma+(\gamma-1)\tau\triangle)/(\gamma\tau)$ with $\triangle = \frac{(1+c(\gamma-1))}{\gamma+c(\gamma-1)}\left(\frac{c-\sigma_0}{ck_0}\right)$, we find from (8) and (9) that, for a characteristic shock for which ζ and η are small, the variables ρ, k, p, etc.have the following asymptotic forms for large t:

$$\rho \sim O(1/t), \quad k \sim O(1/t), \quad p \sim O(1/t), \quad \sigma \sim O(1/t)$$
$$\zeta \sim O(\exp(-\Gamma t)), \quad \text{and} \quad \eta \sim O(\exp(-\Gamma t)) \quad \text{as } t \to \infty,$$

where $\Gamma = -(\delta + \sqrt{\delta^2 + 4\kappa})/2$, showing thereby that the wave decays exponentially; this is again in conformity with the numerical results.

4 Evolution of the weak discontinuity

The evolution of a weak discontinuity for a hyperbolic quasi-linear system of equations satisfying Bernoulli's law has been studied quite extensively in the literature (see, for instance, [8,18,19]). The transport equation for the weak discontinuities across the ith characteristic of a hyperbolic system of n equations of the type (1) is given by (see [20])

$$L_b^{(i,k)}\frac{\mathrm{d}\Lambda_i}{\mathrm{d}t} + L_b^{(i,k)}(U_{bx} + \Lambda_i)(\nabla\lambda^{(i)})_b\Lambda_i + \{(\nabla L^{(i,k)})_b\Lambda_i\}^{\mathrm{tr}}\frac{\mathrm{d}U_b}{\mathrm{d}t}$$
$$+(L_b^{(i,k)}\Lambda_i)((\nabla\lambda^{(i)})_b U_{bx} + \lambda_{bx}{}^{(i)}) - ((\nabla(L^{(i,k)}f)_b)\Lambda_i = 0, \tag{10}$$

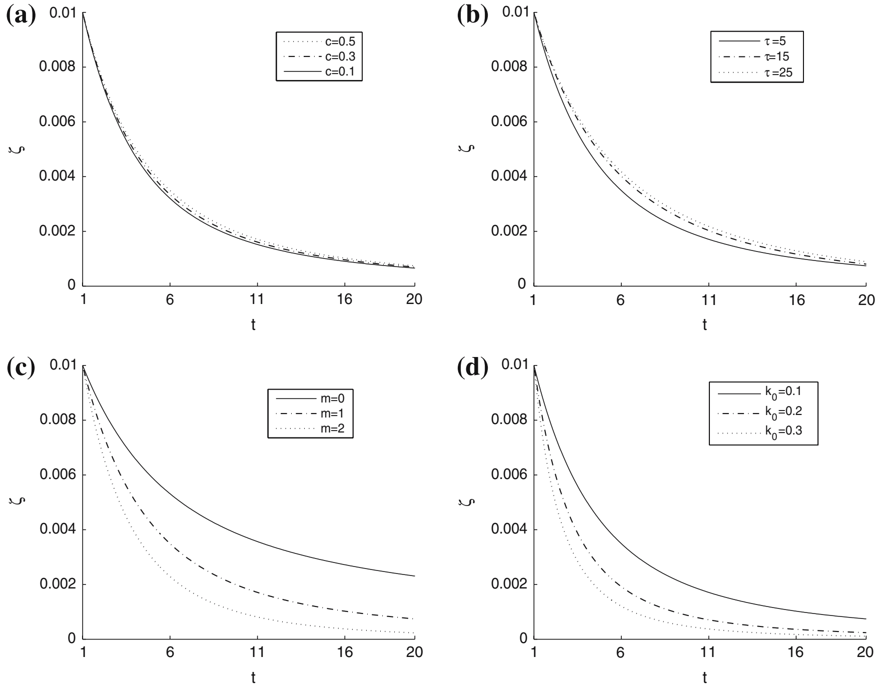

Fig. 1 Variation of ζ with time influenced by the parameters c, τ, m, k_0. (**a**) corresponds to the case when $k_0 = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$; (**b**) uses $k_0 = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $c = 0.001$; (**c**) corresponds to the case $k_0 = 0.1, c = 0.1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$; (**d**) uses $c = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$

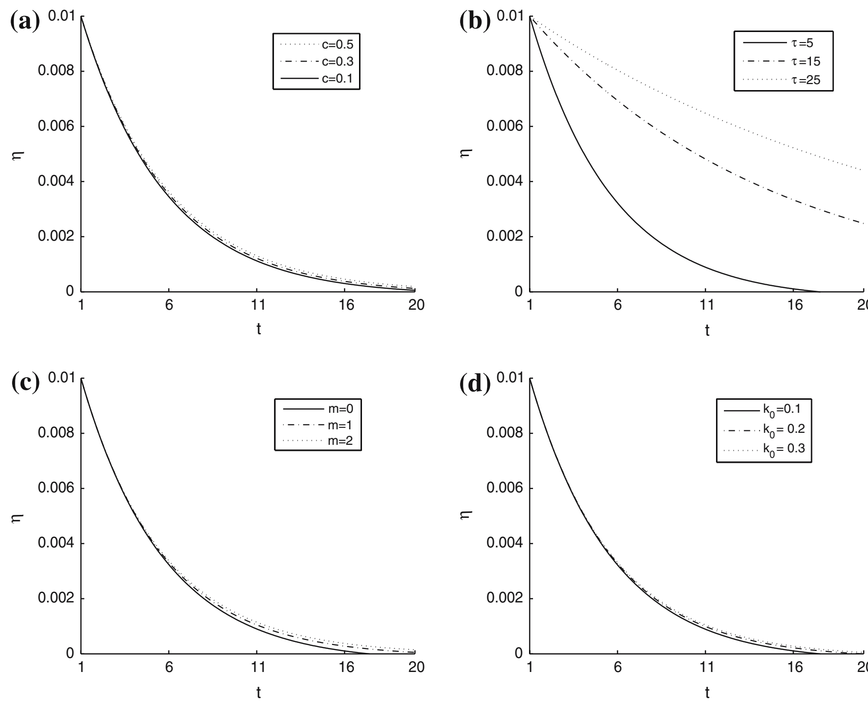

Fig. 2 Variation of η with time influenced by the parameters c, τ, m, k_0. (**a**) corresponds to the case when $k_0 = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$; (**b**) uses $k_0 = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $c = 0.001$; (**c**) corresponds to the case $k_0 = 0.1, c = 0.1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$; (**d**) uses $c = 0.1, m = 1, \sigma_0 = 0.005, \zeta_0 = 0.01$ and $\tau = 5$

where $\lambda^{(i)}$, $i = 1, 2, 3 \ldots p$, are p distinct real eigenvalues of the coefficient matrix A, assumed to be ordered so that $\lambda^{(p)} < \lambda^{(p-1)} < \lambda^{(p-2)} < \cdots < \lambda^{(1)}$, with constant multiplicities m_i such that $\sum_{i=1}^{p} m_i = n$. Here, we denote by $L^{(i,k)}$ and $R^{(i,k)}$, $k = 1, 2, \ldots, m_i$, the left and right eigenvectors of A corresponding to the eigenvalue $\lambda^{(i)}$ and the subscript b refers to the state ahead of the ith characteristic curve at the rear of the characteristic shock, and

$$\Lambda_i = \sum_{k=1}^{m_i} \alpha_k^{(i)}(t) R_b^{(i,k)}, \tag{11}$$

is the jump in U_x across the ith characteristic curve with $\alpha_k^{(i)}$ being the amplitude of the wave propagating along $\mathrm{d}x/\mathrm{d}t = \lambda^{(i)}$. For the system under consideration, let $\Lambda_1 = \alpha^{(1)} R^{(1)}$ denote the jump in U_x across the discontinuity with amplitude $\alpha^{(1)}$, propagating along the curve determined by $\mathrm{d}x/\mathrm{d}t = \lambda^{(1)}$ originating from the point (x_0, t_0) in the region behind the characteristic shock that originates from the point (x_1, t_0) with $x_1 > x_0$. Then, on using (1)–(3) and (11) in (10), we obtain the following transport equation for the incident wave amplitude in terms of the dimensionless variables:

$$\frac{\mathrm{d}\alpha^{(1)}}{\mathrm{d}t} + \frac{(\gamma+1)}{4\rho a}\alpha^{(1)2} + \Theta(t)\alpha^{(1)} = 0, \tag{12}$$

where the wave amplitude $\alpha^{(1)}$ and the wave location x have been non-dimensionalised by their initial values $\alpha_0^{(1)}$ and x_0, respectively, and

$$\Theta = \frac{(\gamma+9)}{4} u_x + \frac{m\gamma u}{2x} + \frac{(5\gamma+1)}{\rho a} p_x - \frac{a\rho_x}{\rho} + \frac{(\gamma-1)}{2a^2\tau}(\sigma_0 - \sigma - c) + \frac{ma}{2x} + \frac{(\gamma-1)c}{2\tau}.$$

Equation (12) yields on integration $\alpha^{(1)} = \alpha_0^{(1)} I(t)\{1 + \alpha_0^{(1)} J(t)\}^{-1}$, where $I(t) = \exp\left(-\int_1^t \Theta(s)\mathrm{d}s\right)$ and $J(t) = \int_1^t \frac{(\gamma+1)}{4a\rho} I(s)\mathrm{d}s$. Using the particular solution described by $(6)_1$, (7) and (8), in dimensionless form, we find that, both I and J are finite and continuous on $[1, \infty)$. Indeed, $I(t) \to 0$ as $t \to \infty$ whereas $J(\infty) < \infty$, implying thereby that, when $\alpha_0^{(1)} > 0$ (i.e., an expansion wave), the amplitude $\alpha^{(1)}$ eventually decays to zero. However for $\alpha_0^{(1)} < 0$ (i.e., a compression wave), there exists a positive quantity $\alpha_c^{(1)}$ defined by $\alpha_c^{(1)} = (|J(\infty)|)^{-1}$ such that for $|\alpha_0^{(1)}| < \alpha_c^{(1)}$, $\alpha^{(1)} \to 0$ as $t \to \infty$, i.e., the wave decays; but for $|\alpha_0^{(1)}| > \alpha_c^{(1)}$, there exist a finite time t_c given by $J(t_c) = 1/|\alpha_0^{(1)}|$ such that $|\alpha^{(1)}| \to \infty$ as $t \to t_c$, showing thereby that a shock appears at $t = t_c$ only when the initial amplitude exceeds a critical value $\alpha_c^{(1)}$. Computed results are in conformity with the analytical results and the corresponding situations are depicted in Figs. 3(a)–3(h), showing that, when $0 < \alpha_c^{(1)} < -\alpha_0^{(1)}$, an increase in any of the variables c, τ^{-1}, k_0 or m delays the onset of a shock (see, Figs. 3(e–h)); however, if the initial amplitude is less than this critical value, i.e., $0 < -\alpha_0^{(1)} < \alpha_c^{(1)}$, a compression wave eventually decays, and the various effects such as relaxation, flow variations behind the shock and geometrical spreading serve to enhance the decay rate (see, Figs. 3(a–d)).

5 Collision of the weak discontinuity with the characteristic shock

The simplest case of a motion, in the $x - t$ plane, involving an interaction of a shock wave with a weak wave can be envisaged when a piston, fitted in a gas-filled container, is instantaneously pushed into the gas column with a uniform velocity giving rise to a shock that races ahead of it. If the piston is suddenly stopped at time, say $t = t_0$, it gives rise to an expansion fan, originating from (x_0, t_0), whose leading front overtakes the shock originating from (x_1, t_0); such a perturbation creates a discontinuity in the acceleration of the shock front besides producing reflected and transmitted waves. For a characteristic shock, this acceleration is not an unknown, and the amplitudes of the reflected and transmitted waves can be completely determined. It is shown in [20] that the general theory for problems of wave interaction, which originated from the work of Jeffrey [21], leads to the results obtained by Brun [22] and Boillat and Ruggeri [23]; the theory has been successfully used for different material media (see [24–26]). In order to study the nature of amplitudes of the reflected and transmitted weak discontinuities, we consider the

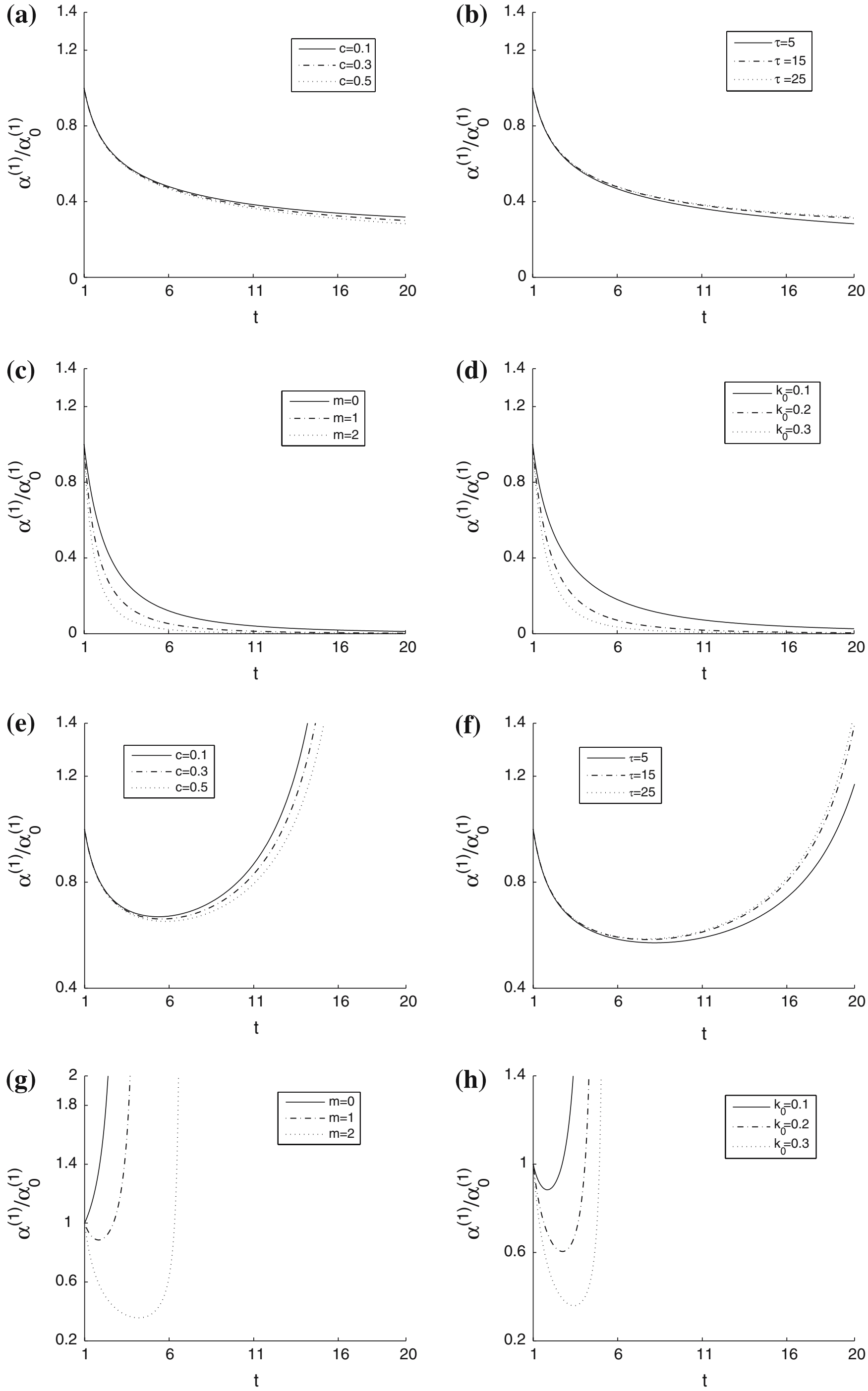

Fig. 3 Variation of the amplitude $\alpha^{(1)}$ with t influenced by the parameters c, τ, k_0 and m when $0 < -\alpha_0^{(1)} < \alpha_c^{(1)}$. (**a**) corresponds to the case when $k_0 = 0.001$, $m = 1$, $\sigma_0 = 0.005$, and $\tau = 5$; (**b**) uses $k_0 = 0.1$, $m = 1$, $\sigma_0 = 0.005$, and $c = 0.3$; (**c**) corresponds to the case $k_0 = 0.1$, $c = 0.1$, $\sigma_0 = 0.005$, and $\tau = 5$; (**d**) uses $c = 0.001$, $m = 1$, $\sigma_0 = 0.005$, and $\tau = 50$. (**e–h**) correspond to the case when $-\alpha_0^{(1)} > \alpha_c^{(1)} > 0$. (**e**) corresponds to the case when $k_0 = 0.001$, $m = 1$, $\sigma_0 = 0.005$, and $\tau = 5$; (**f**) uses $k_0 = 0.1$, $m = 1$, $\sigma_0 = 0.005$, and $c = 0.3$; (**g**) corresponds to the case $k_0 = 0.1$, $c = 0.1$, $\sigma_0 = 0.005$, and $\tau = 5$; (**h**) uses $c = 0.001$, $m = 1$, $\sigma_0 = 0.005$, and $\tau = 50$

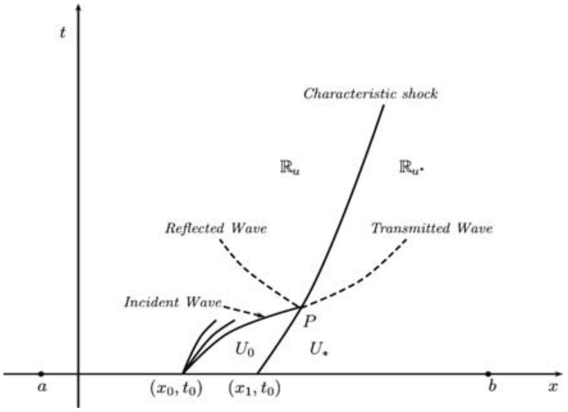

Fig. 4 Interaction of a weak discontinuity with a characteristic shock giving rise to reflected and transmitted waves

generalized conservation systems that are direct consequences of the original system (1), and have the following forms in the regions behind and ahead of the shock (i.e., to the left and to the right of the discontinuity, which propagates with the speed $V = u$):

$$G_t(x,t,U) + F_x(x,t,U) = H(x,t,U),$$
$$G_{*t}(x,t,U_*) + F_{*x}(x,t,U_*) = H_*(x,t,U_*), \tag{13}$$

where U and U_* are the solution vectors behind and ahead of the shock and G, F and H are given by

$$G = \left(\rho, \rho u, \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2, \rho\sigma\right)^{\mathrm{tr}},$$
$$F = \left(\rho u, \rho u^2 + p, u\left(\frac{\gamma p}{(\gamma-1)} + \frac{1}{2}\rho u^2\right), \rho u\sigma\right)^{\mathrm{tr}},$$
$$H = \left(\frac{-m\rho u}{x}, \frac{-m\rho u^2}{x}, -\rho\left\{Q + \frac{mu}{x}\left(\frac{u^2}{2} + \frac{a^2}{(\gamma-1)}\right)\right\}, \rho\left(Q - \frac{mu\sigma}{x}\right)\right)^{\mathrm{tr}}.$$

As in [21], we associate the initial conditions $U(x,t_0) = \phi(x)$ and $U_*(x,t_0) = \phi_*(x)$ defined, respectively, on the adjacent intervals $\mathcal{I}_u$ and $\mathcal{I}_{u_*}$ of the initial line

$$\mathcal{I}_u = \{x | a \le x < x_1; t = t_0\}; \quad \mathcal{I}_{u_*} = \{x |\, x_1 < x \le b; \quad t = t_0\},$$

where the numbers a and b are chosen such that the propagation of a weak discontinuity starting at (x_0, t_0) in $\mathcal{I} = \mathcal{I}_u \cup \mathcal{I}_{u_*}$ is confined to the domain of determinacy $\mathbb{R}$ associated with $\mathcal{I}$ as illustrated in Fig. 4; indeed, U and U_* are continuous in their respective domains of determinacy denoted by $\mathbb{R}_u$ and $\mathbb{R}_{u_*}$, associated with $\mathcal{I}_u$ and $\mathcal{I}_{u_*}$, respectively, but they are discontinuous across the characteristic shock that divides $\mathbb{R}$. We denote by $\phi(x,t) = 0$ the equation of the characteristic in $\mathbb{R}_u$ through (x_0, t_0) forming the initial wave-front trace defined by the equation $\mathrm{d}x/\mathrm{d}t = \lambda^{(1)}$. Let $P(x_p, t_p)$ be the point at which the fastest discontinuity in U, moving along the characteristic $\mathrm{d}x/\mathrm{d}t = \lambda^{(1)}$ and originating from the point (x_0, t_0) intersects the shock, $\mathrm{d}x/\mathrm{d}t = V$. The equation of the new wavefront trace through the point P, defined by the equation $\mathrm{d}x/\mathrm{d}t = \lambda_*^{(1)}$, is denoted by $\phi_*(x,t) = 0$. As in [20], the jumps in U_x across the incident, reflected and transmitted waves at P, which we denote by $\Lambda_1(P)$, $\Lambda_i^{(R)}(P)$ and $\Lambda_i^{*(T)}(P)$, respectively, are given by the relations

$$\Lambda_1(P) = \sum_{k=1}^{m_1} \alpha_k^{(1)}(t_p) R_s^{(1,k)}, \quad \Lambda_i^{(R)}(P) = \sum_{k=1}^{m_i} \alpha_k^{(i)}(t_p) R_s^{(i,k)}, \quad \Lambda_i^{*(T)}(P) = \sum_{k=1}^{m_i^*} \beta_k^{(i)}(t_p) R_s^{*(i,k)}, \tag{14}$$

where a subscript s refers to the values evaluated at the point P on the shock. The evolutionary equations which determine, after interaction, the jump in the shock acceleration $|[\dot{V}]|$, and the amplitudes $\alpha_k^{(i)}$ and $\beta_k^{(i)}$ of reflected and transmitted waves, respectively, are given by the matrix equation

$$
\begin{aligned}
&|[\dot{V}]|(G-G_*)_s+(\nabla G)_s\sum_{i=p-q+1}^{p}\left(\sum_{k=1}^{m_i}\alpha_k^{(i)}\left(V-\lambda^{(i)}\right)^2 R_s^{(i,k)}\right)-(\nabla^* G_*)_s\sum_{j=1}^{q}\left(\sum_{k=1}^{m_j}\beta_k^{(j)}\left(V-\lambda_*^{(j)}\right)^2 R_s^{*(j,k)}\right)\\
&\quad=-(\nabla G)_s\sum_{k=1}^{m_i}\alpha_k^{(1)}\left(V-\lambda^{(1)}\right)^2 R_s^{(1,k)},
\end{aligned}
\tag{15}
$$

which is a system of n algebraic equations. Since at $t=t_p$

$$
\lambda^{(1)}=u+\left(\frac{\gamma p}{\rho}\right)^{1/2},\quad \lambda^{(2)}=u,\quad \lambda^{(3)}=u-\left(\frac{\gamma p}{\rho}\right)^{1/2},
\tag{16}
$$

and

$$
\lambda_*^{(1)}=u+\left(\frac{\gamma p}{\rho-\zeta}\right)^{1/2},\quad \lambda_*^{(2)}=u,\quad \lambda_*^{(3)}=u-\left(\frac{\gamma p}{\rho-\zeta}\right)^{1/2},
\tag{17}
$$

it follows that Lax's evolutionary conditions for a physical shock for an integer l in the interval $1\le l\le p$, (see [27]), namely

$$
\begin{aligned}
&\lambda^{(p)}<\lambda^{(p-1)}<\cdots<\lambda^{(l+1)}<V<\lambda^{(l)}<\cdots<\lambda^{(1)},\\
&\lambda_*^{(p)}<\lambda_*^{(p-1)}<\cdots<\lambda_*^{(l)}<V<\lambda_*^{(l-1)}<\cdots<\lambda_*^{(1)},
\end{aligned}
\tag{18}
$$

imply that, for the system under consideration, we have

$$
\lambda^{(3)}<\lambda^{(2)}=u<\lambda^{(1)},\quad \text{and}\quad \lambda_*^{(3)}<\lambda_*^{(2)}=u<\lambda_*^{(1)}.
\tag{19}
$$

In effect, this asserts that, when the incident wave with velocity $\lambda^{(1)}$ at $t=t_p$ impinges on the characteristic shock, it gives rise to one reflected wave with velocity $\lambda^{(3)}$ and one transmitted wave with velocity $\lambda_*^{(1)}$ along the characteristics issuing from the collision point. The reflected and transmitted-wave amplitudes $\alpha^{(3)}$ and $\beta^{(1)}$, and the jump in shock acceleration $|[\dot{V}]|=\dot{V}_{t_p^+}-\dot{V}_{t_p^-}$ at the collision time $t=t_p$ can be determined from the algebraic system of equations

$$
\begin{aligned}
&(G-G_*)_s|[\dot{V}]|+(\nabla G)_s R_s^{(3)}(V-\lambda_s^{(3)})^2\alpha^{(3)}-(\nabla^* G_*)_s R_s^{*(1)}(V-\lambda_{*s}^{(1)})^2\beta^{(1)}\\
&\quad=-(\nabla G)_s R_s^{(1)}(V-\lambda_s^{(1)})^2\alpha^{(1)}.
\end{aligned}
\tag{20}
$$

In view of (2) and (3), the system (20) can be written as the following system of algebraic equations in the unknowns $|[\dot{V}]|$, $\alpha^{(3)}$ and $\beta^{(1)}$:

$$
\begin{aligned}
&2\zeta\,|[\dot{V}]|+\alpha^{(3)}-\beta^{(1)}=-\alpha^{(1)},\\
&2\zeta u|[\dot{V}]|+(u-a)\alpha^{(3)}-(u+a_*)\beta^{(1)}=-(u-a)\alpha^{(1)},\\
&2\zeta u^2|[\dot{V}]|+\left(u^2-2ua+\tfrac{2a^2}{(\gamma-1)}\right)\alpha^{(3)}-\left(u^2+2ua_*+\tfrac{2a_*{}^2}{(\gamma-1)}\right)\beta^{(1)}=-\left(u^2+2ua+\tfrac{2a^2}{(\gamma-1)}\right)\alpha^{(1)},\\
&2(\rho\sigma_*+\eta\rho_*+\zeta\eta)|[\dot{V}]|+\sigma\alpha^{(3)}-\sigma_*\beta^{(1)}=-\sigma\alpha^{(1)},
\end{aligned}
\tag{21}
$$

On solving, the above algebraic system yields

$$
\beta^{(1)}=\frac{2a^2}{a_*(a_*+a)}\alpha^{(1)},\quad \alpha^{(3)}=\frac{a^2\zeta}{\rho_*(a+a_*)}\alpha^{(1)},\quad |[\dot{V}]|=-\frac{a^2\zeta}{\rho_* a_*(a+a_*)}\alpha^{(1)},
\tag{22}
$$

where $\alpha^{(1)}$ is determined from (12). Equations (22) demonstrate, as would be expected, that in the absence of the incident wave (i.e., $\alpha^{(1)}=0$), the jump in the shock acceleration vanishes and there are no reflected or transmitted waves. Also, it follows from (22) that, if the incident discontinuity is an expansion wave, the reflected and the transmitted waves are also expansive and the shock decelerates after impact; however, if the incident discontinuity is a compression wave, the reflected and transmitted waves are also compressive, and the shock accelerates after impact. These results are in agreement with the observations made by Courant and Friedrichs [28] that if the shock front is overtaken by an expansion (respectively, compression) wave, it is decelerated (respectively, accelerated) and consequently the strength of the shock decreases (respectively, increases).

6 Results and conclusion

In this paper we have been concerned with the evolutionary behaviour of a characteristic shock in a relaxing gas. A wave-interaction problem was examined taking into consideration its interaction with a weak wave. The results of interaction theory were used to study the existence and uniqueness of reflected- and transmitted-wave amplitudes. Evolutionary behaviour of the shock and the jump in the acceleration together with the amplitudes of incident, reflected and transmitted waves, influenced by the relaxation effects, have been studied. Computations of the variables ζ and η, associated with the characteristic shock, have been carried out for the plane ($m = 0$), cylindrically symmetric ($m = 1$) and spherically symmetric ($m = 2$) flows by taking $\gamma = 1.4$, $\sigma_0 = 0.005$, $\zeta_0 = 0.01$, $\eta_0 = 0.01$ and the values of c, τ and k_0 lying in the range $0.001 \leq c \leq 0.5$, $5 \leq \tau \leq 50$ and $0.001 \leq k_0 \leq 0.3$. It is observed that the characteristic shock eventually decays; indeed, an increase in any of the parameters c or τ^{-1} causes ζ and η to decrease, showing thereby that the characteristic shock is progressively smoothened out under the diffusive effects of relaxation present in the medium. We also noted that there is an extra attenuation owing to geometrical spreading of the fluid flow and its variations behind the shock in the sense that an increase in m or k_0 also causes ζ and η to decrease. These numerical results are in conformity with the small- and large-time behaviour of some of the parameters obtained using standard asymptotic analysis; effects of relaxation, geometrical spreading of the fluid flow and its variations behind the shock on the evolutionary behaviour of the incident wave, which overtakes the characteristic shock from behind, have been studied. It is known that an incident compression wave culminates into a shock after a finite time only when its initial amplitude exceeds a critical value, and the effects of relaxation and geometrical spreading of the fluid flow and its variations behind the shock serve to resist the onset of a shock; however, if the initial amplitude of the incident wave is less than the critical value, the wave decays eventually and an increase in any of the parameters c, τ, k_0 or m enhances the decay rate. It was shown that, when the incident wave hits the shock, it gives rise to a reflected wave and a transmitted wave, whose amplitudes are determined along with the jump in the shock acceleration at the collision time. It is observed, as would be expected, that in the absence of the incident wave, neither there is a jump in the shock acceleration nor there exist reflected and transmitted waves. It has been shown that the reflected and transmitted waves are expansive and the shock decelerates after the impact, whenever the incident wave is expansive; however, the reflected and transmitted waves are compressive and the shock accelerates after the impact whenever the incident wave is compressive. These results are in agreement with the observations made by Courant and Friedrichs [28].

Acknowledgements We thank the learned referees for their useful suggestions and comments which made certain points more explicit. Financial support from ISRO-IIT Bombay, Space Technology Cell (Ref. No. 05-IS001) is gratefully acknowledged. Special thanks are due to Mr. M. K. Pandey, a graduate student of the second author, who helped us in the final stages of the revised manuscript.

References

1. Clark JF, McChesney M (1976) Dynamics of relaxing gases. Butterworth, London
2. Scott WA, Johannesen NH (1982) Spherical nonlinear wave propagation in a vibrationally relaxing gas. Proc R Soc London A 382:103–134
3. Sharma VD, Radha Ch (1994) One dimensional planar and nonplanar shock waves in a relaxing gas. Phys Fluids 6:2177–2190
4. Radha Ch, Sharma VD (1995) Propagation and interaction of waves in a relaxing gas. Phil Trans R Soc London A 352:169–195
5. Manickam SAV, Radha Ch, Sharma VD (2003) Far field behaviour of waves in a vibrationally relaxing gas. Appl Numer Math 45:293–307
6. Sharma VD, Srinivasan GK (2005) Wave Interaction in a non equilibrium gas flow. Int J Nonlinear Mech 40:1031–1040
7. Alberto Bressan (2000) Hyperbolic systems of conservation laws: the one-dimensional Cauchy Problem. Oxford University Press UK
8. Boillat G, Ruggeri T (1979) On evolution law of weak discontinuities for hyperbolic quasi-linear systems. Wave Motion 1:149–152
9. Boillat G (1996) Lecture notes in mathematics on recent mathematical methods in nonlinear wave propagation, # 1640. Springer
10. Anile AM, Hunter JK, Pantano P, Russo G (1993) Ray methods for nonlinear waves in fluids and plasma. Longman, UK
11. Serre D (1999) Systems of conservation laws, vol. 1. Cambridge University Press
12. Godlewski E, Raviart P-A (1996) Numerical approximation of hyperbolic systems of conservation laws. Springer

13. Virgopia N, Ferraioli F (1994) On the evolution of characteristic shocks in rotating flows with axial magnetic fields. Continuum Mech Thermodyn 6:31–49
14. Jena J, Sharma VD (1998) On the evolution of a characteristic shock and its Interaction with a discontinuity wave in a relaxing Gas. In: Desanto JA (ed) Fourth international conference on mathematical and numerical aspects of wave propagation, SIAM, 685–687
15. Pert GJ (1980) Self-similar flow with uniform velocity gradient and their use in modelling the free expansion of polytropic gases. J Fluid Mech 100:257–277
16. Sharma VD, Ram R, Sachdev P (1987) Uniformly valid analytical solution to the problem of a decaying shock wave. J Fluid Mech 185:153–170
17. Clarke JF (1978) Small amplitude gasdynamic disturbances in an exploding atmosphere. J Fluid Mech 89:343–355
18. Varley E, Cumberbatch E (1965) Non-linear theory of wavefront propagation. J Inst Math Appl 1:101–112
19. Jeffrey A (1976) Quasilinear hyperbolic systems and waves. Pitman, London
20. Radha Ch, Sharma VD, Jeffrey A (1993) Interaction of shock waves with discontinuities. Appl Anal 50:145–166
21. Jeffrey A (1973) The propagation of weak discontinuities in quasi-linear hyperbolic systems with discontinuous coefficients. Part—I fundamental theory. Appl Anal 3:79–100
22. Brun L (1975) Ondes de choc finies dans les solides elastiques. In: Mandel J, Brun L (ed) Mechanical waves in solids. Springer
23. Boillat G, Ruggeri T (1979) Reflection and transmission of discontinuity waves through a shock wave. General theory including also the case of characteristic shocks. Proc R Soc Edinburgh 83A:17–24
24. Ruggeri T (1980) Interaction between a discontinuity wave and a shock wave; critical time for the fastest transmitted wave; example of a polytropic fluid. Appl Anal 11:103–112
25. Jena J (2005) Group theoretic method for analysing interaction of a weak discontinuity wave with a bore in shallow water waves. Appl Anal 84:37–48
26. Jena J (2007) Lie-group theoretic method for analyzing interaction of discontinuous waves in a relaxing gas. ZAMP 58:416–430
27. Lax PD (1957) Hyperbolic system of conservation laws II. Commun Pure Appl Math 10:537–566
28. Courant R, Friedrichs KO (1999) Supersonic flow and shock waves. Springer, New York